A note on uniqueness of parameter identification in a jump diffusion model

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Abstract

In this note, we consider an inverse problem in a jump diffusion model. Using characteristic functions we prove the injectivity of the forward operator mapping the five parameters determining the model to the density function of the return distribution.

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1 Forward operator in a jump diffusion model

Let the random price process $(S_t; t \in [0, \infty))$ of a financial asset be described by a jump diffusion process. More precisely, we assume that the price process follows the stochastic differential equation

$$dS_t = S_t((\mu - \lambda\nu)dt + \sigma dW_t) + S_{t-} dN_t^c, \quad t \in (0, \infty),\ S_0 = \xi,$$

where $(W_t; t \in [0, \infty))$ is a standard Wiener process and $(N_t^c; t \in [0, \infty))$ an independent compound Poisson process with jump amplitudes $(Y_j - 1; j \in \mathbb{N})$. We assume that the random variables $\ln Y_j, j \in \mathbb{N}$, are independent Gaussian variables with mean μ_Y and variance σ_Y^2. The parameter $\lambda \geq 0$ expresses the intensity of the underlying Poisson process $(N_t; t \in [0, \infty))$. Furthermore we have $\nu = \mathrm{e}^{\mu_Y + \frac{1}{2}\sigma_Y^2} - 1$, $\mu \in \mathbb{R}, \sigma^2 > 0$ and an initial value ξ. The model has been considered in detail in [1].

Then it holds for the logarithmic returns $r_t = \ln\left(\frac{S_t}{S_0}\right)$ and fixed lag $t > 0$

$$r_t = \tilde{\mu}t + \sigma W_t + \sum_{j=1}^{N_t} \ln Y_j$$

with parameter $\tilde{\mu} = \mu - \frac{1}{2}\sigma^2 - \lambda\nu$. The independence of the random processes $(W_t; t \in [0, \infty))$, $(N_t; t \in [0, \infty))$ and the random variables $(Y_j - 1; j \in \mathbb{N})$ as well as the distribution assumptions $W_t \sim N(0, t),\ \ln Y_j \sim N(\mu_Y, \sigma_Y^2),\ N_t \sim Poisson(\lambda t)$ allow to calculate the distribution function

$$F_{r_t}(x, \underline{p}) = \sum_{j=0}^{\infty} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^j}{j!} \Phi\left(\frac{x - (\tilde{\mu}t + j\mu_Y)}{\sqrt{\sigma^2 t + j\sigma_Y^2}}\right), \quad x \in \mathbb{R}, \tag{1.1}$$

for the returns r_t and the associated probability density function

$$f_{r_t}(x, \underline{p}) = \sum_{j=0}^{\infty} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^j}{j!\sqrt{\sigma^2 t + j\sigma_Y^2}} \phi\left(\frac{x - (\tilde{\mu}t + j\mu_Y)}{\sqrt{\sigma^2 t + j\sigma_Y^2}}\right), \quad x \in \mathbb{R}, \tag{1.2}$$

as well as the characteristic function

$$\varphi_{r_t}(\theta, \underline{p}) = \exp\left(i\tilde{\mu}t\theta - \frac{\sigma^2 t}{2}\theta^2 + \lambda t\left(\exp\left(i\mu_Y\theta - \frac{\sigma_Y^2}{2}\theta^2\right) - 1\right)\right), \quad \theta \in \mathbb{R}. \tag{1.3}$$

In the formulae (1.1) – (1.3) the corresponding functions explicitly depend on the parameter vector

$$\underline{p} = (\mu, \sigma, \lambda, \mu_Y, \sigma_Y)$$

determining the model under consideration.

We are going to show that the inverse problem of identifying $\underline{p}$ from given density function f_{r_t} is, for fixed $t > 0$, uniquely solvable on some domain whenever a solution exists.

In this context, we consider the forward operator

$$\mathrm{F}: \quad \underline{p} \in D \quad \mapsto \quad f_{r_t}$$

defined on the domain

$$D = \{\underline{p} = (\mu, \sigma, \lambda, \mu_Y, \sigma_Y): \ \mu \in \mathbb{R}, \ \sigma > 0, \ \lambda \geq 0, \ \mu_Y \in \mathbb{R}, \ \sigma_Y \geq 0\}, \tag{1.4}$$

where the nonlinear operator F is defined by formula (1.2).

Obviously, in the case $\lambda = 0$ no jumps occur, i.e., the values $\mu_Y \in \mathbb{R}$ and $\sigma_Y \geq 0$ may be arbitrary and do not influence the value or the distribution of the returns. Similarly, for $\mu_Y = \sigma_Y = 0$, all jumps are of zero height. Then they also do not influence the value or the distribution of the returns, and the parameter $\lambda \geq 0$ may be arbitrary. In [1] it was conjectured that the following theorem is true.

Theorem 1.1
The operator $\mathrm{F} : \underline{p} \in D \mapsto f_{r_t}$ *defined by formula (1.2) is injective on the restricted domain*

$$\hat{D} = \{\underline{p} \in D : \ \lambda \left(\sigma_Y^2 + \mu_Y^2\right) \ \neq \ 0\}. \tag{1.5}$$

We prove theorem 1.1 in the subsequent paragraph setting for simplicity $t = 1$.

2 Proof of the theorem

Probability distributions of real valued random variables are uniquely determined by the corresponding characteristic functions (cf., e.g., [2]). Therefore we will show that for equal characteristic functions of returns either the no-jump-condition $\lambda \left(\sigma_Y^2 + \mu_Y^2\right) = 0$ (excluded in (1.5)) is valid or all parameters coincide.

Let us assume that for two parameter vectors

$$^1\underline{p} = (^1\mu, {}^1\sigma, {}^1\lambda, {}^1\mu_Y, {}^1\sigma_Y) \in D \qquad \text{and} \qquad ^2\underline{p} = (^2\mu, {}^2\sigma, {}^2\lambda, {}^2\mu_Y, {}^2\sigma_Y) \in D$$

with D from (1.4) the distributions of the returns ^1r_t and ^2r_t with $t = 1$ and hence the corresponding characteristic functions

$$\varphi_{^1r_1}\left(\theta, {}^1\underline{p}\right) = \exp\left(i\,{}^1\tilde{\mu}\theta - \frac{^1\sigma^2}{2}\theta^2 + {}^1\lambda\left(\exp\left(i\,{}^1\mu_Y\theta - \frac{^1\sigma_Y^2}{2}\theta^2\right) - 1\right)\right), \quad \theta \in \mathbb{R},$$
$$\varphi_{^2r_1}\left(\theta, {}^2\underline{p}\right) = \exp\left(i\,{}^2\tilde{\mu}\theta - \frac{^2\sigma^2}{2}\theta^2 + {}^2\lambda\left(\exp\left(i\,{}^2\mu_Y\theta - \frac{^2\sigma_Y^2}{2}\theta^2\right) - 1\right)\right), \quad \theta \in \mathbb{R},$$

coincide. Then also the logarithms of the characteristic functions coincide, i.e.,

$$\begin{aligned} i\,{}^1\tilde{\mu}\theta - \frac{{}^1\sigma^2}{2}\theta^2 + {}^1\lambda\left(\exp\left(i\,{}^1\mu_Y\theta - \frac{{}^1\sigma_Y^2}{2}\theta^2\right) - 1\right) \\ = i\,{}^2\tilde{\mu}\theta - \frac{{}^2\sigma^2}{2}\theta^2 + {}^2\lambda\left(\exp\left(i\,{}^2\mu_Y\theta - \frac{{}^2\sigma_Y^2}{2}\theta^2\right) - 1\right), \qquad \forall\theta \in \mathbb{R}. \end{aligned}$$

So we have for all real numbers $\theta \in \mathbb{R}$

$$ {}^2\lambda - {}^1\lambda + i\,({}^1\tilde{\mu} - {}^2\tilde{\mu})\theta - \frac{{}^1\sigma^2 - {}^2\sigma^2}{2}\theta^2 = {}^2\lambda \exp\left(i\,{}^2\mu_Y\theta - \frac{{}^2\sigma_Y^2}{2}\theta^2\right) - {}^1\lambda \exp\left(i\,{}^1\mu_Y\theta - \frac{{}^1\sigma_Y^2}{2}\theta^2\right). \tag{2.1}$$

The left hand side of this identity is a second order polynomial in θ, hence the third derivative with respect to θ vanishes. The third order derivative of the right hand side is

$$ {}^2A(\theta)\exp\left({}^2B(\theta)\right) - {}^1A(\theta)\exp\left({}^1B(\theta)\right)$$

with polynomials

$$\begin{aligned} {}^1A(\theta) &= {}^1\lambda\left(-3{}^1\sigma_Y^2 + (i\,{}^1\mu_Y - {}^1\sigma_Y^2\theta)^2\right)(i\,{}^1\mu_Y - {}^1\sigma_Y^2\theta), \\ {}^2A(\theta) &= {}^2\lambda\left(-3{}^2\sigma_Y^2 + (i\,{}^2\mu_Y - {}^2\sigma_Y^2\theta)^2\right)(i\,{}^2\mu_Y - {}^2\sigma_Y^2\theta), \\ {}^1B(\theta) &= i\,{}^1\mu_Y\theta - \frac{{}^1\sigma_Y^2}{2}\theta^2, \\ {}^2B(\theta) &= i\,{}^2\mu_Y\theta - \frac{{}^2\sigma_Y^2}{2}\theta^2. \end{aligned}$$

So we get the relation

$$ {}^2A(\theta)\exp\left({}^2B(\theta)\right) - {}^1A(\theta)\exp\left({}^1B(\theta)\right) = 0, \qquad \forall\,\theta \in \mathbb{R}. \tag{2.2}$$

If one of the polynomials ${}^1A(\theta), {}^2A(\theta)$ vanishes identically, say ${}^1A(\theta)$, which is possible if and only if ${}^1\lambda = 0$ or ${}^1\mu_Y = {}^1\sigma_Y^2 = 0$, we get ${}^1\lambda\,({}^1\sigma_Y^2 + {}^1\mu_Y^2) = 0$ and consequently

$$ {}^2A(\theta)\exp\left({}^2B(\theta)\right) = 0, \qquad \forall\,\theta \in \mathbb{R}.$$

Then ${}^2A(\theta)$ also vanishes identically and the no-jump-condition

$$ {}^2\lambda\left({}^2\sigma_Y^2 + {}^2\mu_Y^2\right) = 0 \qquad \text{equivalent to} \qquad {}^2\lambda = 0 \ \text{ or } \ {}^2\mu_Y = {}^2\sigma_Y^2 = 0$$

must be fulfilled.

If ${}^1\sigma_Y = 0$ and ${}^1\lambda{}^1\mu_Y \neq 0$, then we have a nonzero constant ${}^1A(\theta) \equiv -i\,{}^1\lambda{}^1\mu_Y^3$ and ${}^2A(\theta)$ must be the same constant yielding ${}^2\sigma_Y = 0$. This implies ${}^1B(\theta) = i\,{}^1\mu_Y\theta \equiv {}^2B(\theta) = i\,{}^2\mu_Y\theta$ as well as ${}^1\mu_Y = {}^2\mu_Y$ and ${}^1\lambda = {}^2\lambda$.

If otherwise both polynomials

$$ {}^1A(\theta) = -{}^1\lambda\ {}^1\sigma_Y^6\left(\theta - \frac{i\,{}^1\mu_Y}{{}^1\sigma_Y^2} - \frac{\sqrt{3}}{{}^1\sigma_Y}\right)\left(\theta - \frac{i\,{}^1\mu_Y}{{}^1\sigma_Y^2} + \frac{\sqrt{3}}{{}^1\sigma_Y}\right)\left(\theta - \frac{i\,{}^1\mu_Y}{{}^1\sigma_Y^2}\right)$$

and

$$^2A(\theta) = -^2\lambda\ ^2\sigma_Y^6 \left(\theta - \frac{i\,^2\mu_Y}{^2\sigma_Y^2} - \frac{\sqrt{3}}{^2\sigma_Y}\right)\left(\theta - \frac{i\,^2\mu_Y}{^2\sigma_Y^2} + \frac{\sqrt{3}}{^2\sigma_Y}\right)\left(\theta - \frac{i\,^2\mu_Y}{^2\sigma_Y^2}\right)$$

do not vanish identically, their zeros must coincide due to fact that

$$\left|\exp\left(^1B(\theta)\right)\right| > 0, \quad \left|\exp\left(^2B(\theta)\right)\right| > 0, \qquad \forall\,\theta \in \mathbb{R}.$$

So we get $^2A(\theta) = c\,^1A(\theta)$ with a nonzero constant c.

If $^1\mu_Y = 0$ and $^1\lambda\,^1\sigma_Y \neq 0$, then we have $^2A(0) = {}^1A(0) = 0$ and hence $^2\mu_Y = 0$. This implies $^1\sigma_Y = {}^2\sigma_Y$ because the zeros coincide. Thus we have $^2B(\theta) = {}^1B(\theta)$ and with (2.2) the equation $^1\lambda = {}^2\lambda$.

In all other cases we set $\theta = 0$ in formula (2.2) and conclude $c = 1$, hence $^2A(\theta) = {}^1A(\theta)$ for all $\theta \in \mathbb{R}$. This leads to the relations

$$\frac{i\,^1\mu_Y}{^1\sigma_Y^2} = \frac{i\,^2\mu_Y}{^2\sigma_Y^2},$$

$$\frac{i\,^1\mu_Y}{^1\sigma_Y^2} - \frac{\sqrt{3}}{^1\sigma_Y} = \frac{i\,^2\mu_Y}{^2\sigma_Y^2} - \frac{\sqrt{3}}{^2\sigma_Y},$$

$$-^1\lambda\ ^1\sigma_Y^6 = -^2\lambda\ ^2\sigma_Y^6$$

from which the identities

$$^1\sigma_Y = {}^2\sigma_Y, \quad ^1\mu_Y = {}^2\mu_Y, \quad \text{and} \quad ^1\lambda = {}^2\lambda$$

follow.

This yields for equation (2.1)

$$i\,(^1\tilde{\mu} - {}^2\tilde{\mu})\theta - \frac{^1\sigma^2 - {}^2\sigma^2}{2}\theta^2 = 0 \qquad \forall\,\theta \in \mathbb{R},$$

consequently

$$^1\tilde{\mu} = {}^2\tilde{\mu} \ \text{ and } \ ^1\sigma = {}^2\sigma.$$

Then the equations

$$^1\tilde{\mu} = {}^1\mu - \frac{^1\sigma^2}{2} - {}^1\lambda\left(\mathrm{e}^{^1\mu_Y + \frac{1}{2}\,^1\sigma_Y^2} - 1\right), \quad ^2\tilde{\mu} = {}^2\mu - \frac{^2\sigma^2}{2} - {}^2\lambda\left(\mathrm{e}^{^2\mu_Y + \frac{1}{2}\,^2\sigma_Y^2} - 1\right)$$

lead to

$$^1\mu = {}^2\mu.$$

This proves theorem 1.1.

Note that for a forward operator $\mathrm{F} : \underline{p} \mapsto F_{r_t}$ (see formula (1.1)) the injectivity can also be shown for an extended domain including the case $\sigma = 0$.

References

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